

ON THE RELATION OF SPECIAL LINEAR ALGEBRAIC COBORDISM TO WITT GROUPS.

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ABSTRACT. We reconstruct derived Witt groups via special linear algebraic cobordism. There is a morphism $\varphi: MSL^{*,*} \rightarrow W^*$ of ring cohomology theories which sends the canonical Thom class th^{MSL} to the Thom class th^W . We show that for every smooth variety X this morphism induces an isomorphism

$$\overline{\varphi}: MSL^{*,*}(X)/(\eta - 1) \otimes_{MSL^{4*,2*}(pt)} W^{2*}(pt) \rightarrow W^*(X),$$

where η is the stable Hopf map. This result is an analogue of the result by Panin and Walter reconstructing hermitian K -theory using symplectic algebraic cobordism.

1. INTRODUCTION.

The main result of this paper relates special linear algebraic cobordism to derived Witt groups. It is a variation of the algebraic version of Conner and Floyd's theorem [CF66, Theorem 10.2] that reconstructs real K -theory using symplectic cobordism. The direct algebraic analogue involving symplectic algebraic cobordism and hermitian K -theory was obtained by Panin and Walter [PW10d]. It claims that for every smooth variety X there exists a natural isomorphism

$$MSp^{*,*}(X) \otimes_{MSp^{4*,2*}(pt)} KO_0^{[2*]}(pt) \xrightarrow{\cong} KO_*^{[*]}(X).$$

Here $KO_*^{[*]}(-)$ are Schlichting's hermitian K -theory groups [Sch10a, Sch12] and $MSp^{*,*}(-)$ stands for the symplectic algebraic cobordisms that is the ring cohomology theory represented in the motivic stable homotopy category $\mathcal{SH}(k)$ by the spectrum MSp [PW10c]. Symplectic algebraic cobordism is the universal symplectically oriented cohomology theory [PW10c, Theorem 13.2] and the above isomorphism is induced by the homomorphism $MSp^{*,*}(X) \rightarrow KO_*^{[*]}(X)$ arising from the symplectic orientation of hermitian K -theory.

In fact, hermitian K -theory is not only symplectically oriented but also SL -oriented cohomology theory, thus by the universality of special linear algebraic cobordisms [PW10c, Theorem 5.9] there is a natural morphism $MSL^{*,*}(X) \rightarrow KO_*^{[*]}(X)$ and one may ask whether we can substitute in the above theorem the special linear cobordisms for the symplectic ones. In this paper we show that after inverting a certain element $\eta \in MSL^{-1,-1}(pt)$ and the corresponding element in $KO_{-1}^{[-1]}(pt)$ one indeed obtains an isomorphism. The element η arises in the following way. Recall that there is a spectrum BO representing hermitian K -theory [PW10b], i.e. for this spectrum one has natural isomorphisms

$BO^{p,q}(X/U) \cong KO_{2q-p}^{[q]}(X, U)$. Every represented cohomology theory is a module over stable cohomotopy groups $\pi^{*,*}(pt)$ in a natural way. Hence for every motivic space Y we have a structure of $\pi^{*,*}(pt)$ -module on $MSL^{*,*}(Y)$ and on $BO^{*,*}(Y)$ and can localize these modules in the stable Hopf map $\eta \in \pi^{-1,-1}(pt)$ corresponding to the Σ_T^∞ -suspension of $H: \mathbb{A}^2 - \{0\} \rightarrow \mathbb{P}^1$, $H(x, y) = [x, y]$. Theorem 8 claims that for every small pointed motivic space Y there exists a natural isomorphism

$$MSL_\eta^{*,*}(Y) \otimes_{MSL^{4*,2*}(pt)} BO_\eta^{4*,2*}(pt) \xrightarrow{\sim} BO_\eta^{*,*}(Y).$$

See Definition 14 for the precise sense of $BO_\eta^{4*,2*}(pt)$.

For every smooth variety X the corresponding pointed motivic space X_+ is small. Moreover, there are natural isomorphisms

$$BO_\eta^{*,*}(X) \cong W^*(X)[\eta, \eta^{-1}],$$

with an appropriate grading ring structure on the right-hand side, see Theorem 5. Here $W^*(X)$ stands for the derived Witt groups defined by Balmer [Bal99], one can find a comprehensive survey including definitions and applications of these groups in [Bal05]. Thus Theorem 8 yields that for every smooth variety X there exists a natural isomorphism

$$MSL^{*,*}(X)/(\eta - 1) \otimes_{MSL^{4*,2*}(pt)} W^{2*}(pt) \xrightarrow{\sim} W^*(X).$$

The paper is organized as follows. In section 2 we recall the general context of unstable $H_\bullet(k)$ and stable $\mathcal{SH}(k)$ motivic homotopy categories introduced by Morel and Voevodsky [MV99, Voe98]. Then we do some preliminary calculations with stable cohomotopy groups $\pi^{*,*}$ staying mainly in the unstable homotopy category $H_\bullet(k)$. In section 4 we deal with special linear and symplectic orientations and recall the universality theorems for the algebraic cobordisms MSL and MSp . In the next two sections we deal with hermitian K -theory and the spectrum BO , in particular we show that $BO_\eta^{*,*}(X)$ is isomorphic to the Laurent polynomial ring over the derived Witt groups $W^*(X)$.

The last section is devoted to the main theorem relating MSL -cobordism to the derived Witt groups. Using the Panin and Walter's result one can easily show that the examined homomorphism is surjective and construct a section. The main issue is to show that the section maps Thom classes of the special linear bundles to the corresponding Thom classes. It is an easy observation that the claim holds for the Thom classes of symplectic bundles and, thanks to the theory of characteristic classes developed in [An12], the general case follows from this observation. Recall that for an SL -oriented cohomology theory one has a good theory of characteristic classes only after inverting the stable Hopf map η , for the details see loc.cit. In this case one has analogues of the projective bundle theorem and the splitting principle. Roughly speaking, the splitting principle claims that working with SL -oriented cohomology theories with inverted stable Hopf map one may think that every special linear bundle of even rank is a direct sum of special linear bundles of rank two. But a special linear bundle of rank two is a symplectic bundle in a natural way, thus every special linear bundle

of even rank is, in a certain sense, a symplectic bundle. See [An12, Theorem 7] for the precise statement.

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2. PRELIMINARIES ON $\mathcal{SH}(k)$ AND RING COHOMOLOGY THEORIES.

Throughout this paper k is a field of characteristic different from 2.

Let Sm/k be the category of smooth varieties over k . A motivic space over k is a simplicial presheaf on Sm/k . Each variety $X \in Sm/k$ defines a motivic space $Hom_{Sm/k}(-, X)$ constant in the simplicial direction. We write pt for the $\text{Spec } k$ regarded as a motivic space. Inverting the weak motivic equivalences in the category of pointed motivic spaces gives the pointed motivic unstable homotopy category $H_\bullet(k)$.

Let $T = \mathbb{A}^1/(A^1 - \{0\})$ be the Morel-Voevodsky object. A T -spectrum M [Jar00] is a sequence of pointed motivic spaces (M_0, M_1, M_2, \dots) equipped with the maps $\sigma_n: T \wedge M_n \rightarrow M_{n+1}$. A map of T -spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps. Inverting the stable motivic weak equivalences as in [Jar00] gives the motivic stable homotopy category $\mathcal{SH}(k)$.

A pointed motivic space X gives rise to a suspension T -spectrum $\Sigma_T^\infty X$. Set $\mathbb{S} = \Sigma_T^\infty(pt_+)$ for the spherical spectrum. Both $H_\bullet(k)$ and $\mathcal{SH}(k)$ are equipped with symmetric monoidal structures (\wedge, pt_+) and (\wedge, \mathbb{S}) respectively and

$$\Sigma_T^\infty: H_\bullet(k) \rightarrow \mathcal{SH}(k)$$

is a strict symmetric monoidal functor.

Recall that there are two spheres in $H_\bullet(k)$, the simplicial one $S^{1,0} = S_s^1 = \Delta^1/\partial(\Delta^1)$ and $S^{1,1} = (\mathbb{G}_m, 1)$. For the integers $p, q \geq 0$ we write $S^{p+q,q}$ for $(\mathbb{G}_m, 1)^{\wedge q} \wedge (S_s^1)^{\wedge p}$ and $\Sigma^{p+q,q}$ for the suspension functor $-\wedge S^{p+q,q}$. This functor becomes invertible in the stable homotopy category $\mathcal{SH}(k)$, so we extend the notation to arbitrary integers p, q in an obvious way.

Any T -spectrum A defines a bigraded cohomology theory on the category of pointed motivic spaces. Namely, for a pointed space (X, x) one sets

$$A^{p,q}(X, x) = Hom_{\mathcal{SH}(k)}(\Sigma^\infty(X, x), \Sigma^{p,q}A)$$

and $A^{*,*}(X, x) = \bigoplus_{p,q} A^{p,q}(X, x)$. In case of $i - j, j \geq 0$ one has a canonical suspension isomorphism $A^{p,q}(X, x) \cong A^{p+i,q+j}(\Sigma^{i,j}(X, x))$ induced by the shuffling isomorphism $S^{p,q} \wedge S^{i,j} \cong S^{p+i,q+j}$. In the motivic homotopy category there is a canonical isomorphism $T \cong S^{2,1}$, we put

$$\Sigma_T: A^{*,*}(X) \xrightarrow{\cong} A^{*+2,*+1}(X \wedge T)$$

for the corresponding suspension isomorphism. See Definition 4 for the details.

For a variety X we set $A^{p,q}(X) = A^{p,q}(X_+, +)$ for an externally pointed motivic space $(X_+, +)$ associated to X . Groups $A^{*,*}(X)$ are defined accordingly. Set $\pi^{i,j}(X) = \mathbb{S}^{i,j}(X)$ to be the stable cohomotopy groups of X . Given a closed embedding $i: Z \rightarrow X$ of varieties we write $Th(i)$ for $X/(X - Z)$. For a vector bundle $E \rightarrow X$ set $Th(E) = E/(E - X)$ to be the Thom space of E .

A commutative ring T -spectrum is a commutative monoid (A, m, e) in $(\mathcal{SH}(k), \wedge, \mathbb{S})$. The cohomology theory defined by such spectrum is a ring cohomology theory satisfying a certain bigraded commutativity condition described by Morel. Recall some facts related to the multiplicative structure.

(1) *Cup-product*: for a pointed motivic space Y there is a functorial graded ring structure

$$\cup: A^{*,*}(Y) \times A^{*,*}(Y) \rightarrow A^{*,*}(Y).$$

Moreover, let $i_1: Z_1 \rightarrow X$ and $i_2: Z_2 \rightarrow X$ be closed embeddings and put $i_{12}: Z_1 \cap Z_2 \rightarrow X$. Then there is a functorial, bilinear and associative cup-product

$$\cup: A^{*,*}(Th(i_1)) \times A^{*,*}(Th(i_2)) \rightarrow A^{*,*}(Th(i_{12})).$$

In particular, setting $Z_1 = X$ we obtain an $A^{*,*}(X)$ -module structure on $A^{*,*}(Th(i_2))$. We will sometimes omit \cup from the notation.

(2) *Cross-product*: let Y_1, Y_2 be pointed motivic spaces. Then there exists a functorial bilinear pairing

$$\times: A^{*,*}(Y_1) \times A^{*,*}(Y_2) \rightarrow A^{*,*}(Y_1 \wedge Y_2).$$

For a pointed motivic space Y let $\Delta: Y \rightarrow Y \wedge Y$ be the diagonal morphism. There is the following identity relating cross-product to the cup-product:

$$a \cup b = \Delta^A(a \times b).$$

(3) *Module structure over stable cohomotopy groups*: for every motivic space Y we have a homomorphism of graded rings $\pi^{*,*}(Y) \rightarrow A^{*,*}(Y)$, which defines a $\pi^{*,*}(pt)$ -module structure on $A^{*,*}(Y)$. For a smooth variety X the ring $A^{*,*}(X)$ is a graded $\pi^{*,*}(pt)$ -algebra via $\pi^{*,*}(pt) \rightarrow \pi^{*,*}(X) \rightarrow A^{*,*}(X)$.

(4) *Graded ϵ -commutativity* [Mor00]: let $\epsilon \in \pi^{0,0}(pt)$ be the element corresponding under the suspension isomorphism to the morphism $T \rightarrow T, x \mapsto -x$. Then for every motivic space X and $a \in A^{i,j}(X), b \in A^{p,q}(X)$ we have

$$ab = (-1)^{ip} \epsilon^{jq} ba.$$

Recall that $\epsilon^2 = 1$.

3. MOTIVIC SPHERES.

In this section we recall certain canonical isomorphisms in the homotopy category and the definition of the stable Hopf map. Then we carry out a number of computations in the homotopy category in order to present in a convenient way the connecting homomorphism in the localization sequence for the embedding $\{0\} \subset \mathbb{A}^1$. The goal of this section is to show that one can obtain the stable Hopf map applying this connecting homomorphism to a certain natural element of the cohomotopy groups. Throughout this section we will write $\mathbb{G}_m, \mathbb{A}^1, \mathbb{P}^1$ for

the corresponding motivic spaces pointed by 1 and $\mathbb{G}_{m+}, \mathbb{A}_+^1, \mathbb{P}_+^1$ for the motivic spaces pointed externally.

In order to write down the canonical isomorphisms for the different models of the motivic spheres one needs the cone construction.

Definition 1. Let $i: X \rightarrow Y$ be a morphism of motivic spaces. The space $Cone(i)$ defined via the cocartesian square

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow in_1 & & \downarrow \\ X \wedge \Delta^1 & \longrightarrow & Cone(i) \end{array}$$

is called the *cone of the morphism i* .

The following isomorphisms are well-known, see [MV99, Lemma 2.15, Example 2.20].

Definition 2. Set $\rho = \rho_2 \circ \rho_1^{-1}: T \xrightarrow{\cong} \mathbb{G}_m \wedge S_s^1 = S^{2,1}$ for the canonical isomorphism in the homotopy category defined via

$$T \xleftarrow{\rho_1} Cone(i) \xrightarrow{\rho_2} \mathbb{G}_m \wedge S_s^1$$

where i stands for the natural embedding $\mathbb{G}_m \rightarrow \mathbb{A}^1$ and the isomorphisms ρ_1 and ρ_2 and induced by the maps $\Delta^1 \rightarrow pt$ and $\mathbb{A}^1 \rightarrow pt$ respectively.

Definition 3. Set $\sigma = \sigma_2^{-1} \circ \sigma_1: (\mathbb{A}^2 - \{0\}, (1, 1)) \xrightarrow{\cong} \mathbb{G}_m \wedge T$ for the canonical isomorphism in the homotopy category. It is defined via

$$(\mathbb{A}^2 - \{0\}, (1, 1)) \xrightarrow{\sigma_1} (\mathbb{A}^2 - \{0\}) / ((\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1)) \xleftarrow{\sigma_2} \mathbb{G}_m \wedge T$$

where σ_1 is induced by the identity map on $\mathbb{A}^2 - \{0\}$ and σ_2 is induced by the natural embedding $\mathbb{G}_m \times \mathbb{A}^1 \subset \mathbb{A}^2 - \{0\}$. Recall that σ_1 is an isomorphism since $(\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1)$ is \mathbb{A}^1 -contractible, while σ_2 is induced by the excision isomorphism $\mathbb{G}_{m+} \wedge T \cong (\mathbb{A}^2 - \{0\}) / (\mathbb{A}^1 \times \mathbb{G}_m)$, so it is an isomorphism as well.

Definition 4. Identifying $T \cong S^{2,1}$ via ρ set

$$\Sigma_T = (id \wedge \rho^\pi) \circ \Sigma^{2,1}: \pi^{p,q}(X) \xrightarrow{\cong} \pi^{p+2,q+1}(X \wedge T)$$

for the suspension isomorphism.

Definition 5. The *Hopf map* is the canonical morphism of the varieties

$$H: \mathbb{A}^2 - \{0\} \rightarrow \mathbb{P}^1$$

defined via $H(x, y) = [x, y]$. Pointing $\mathbb{A}^2 - \{0\}$ by $(1, 1)$ and \mathbb{P}^1 by $[1 : 1]$ and taking the suspension spectra we obtain the corresponding morphism

$$\Sigma_T^\infty H \in Hom_{\mathcal{SH}(k)}(\Sigma^\infty(\mathbb{A}^2 - \{0\}, (1, 1)), \Sigma^\infty(\mathbb{P}^1, [1 : 1])).$$

In order to interpret this morphism as an element of $\pi^{-1,-1}(pt)$ we introduce the following isomorphism. Let $\vartheta = \rho\vartheta_2^{-1}\vartheta_1$ be the composition

$$\vartheta: (\mathbb{P}^1, [1 : 1]) \xrightarrow{\vartheta_1} \mathbb{P}^1 / \mathbb{A}^1 \xleftarrow{\vartheta_2} T \xrightarrow{\rho} S^{2,1}.$$

Here ϑ_1 is induced by the identity map on \mathbb{P}^1 and ϑ_2 is the excision isomorphism given by $\vartheta_2(x) = [x : 1]$. The *stable Hopf map* is the unique element

$$\begin{aligned} \eta \in \pi^{-1,-1}(pt) &\cong \pi^{2,1}((\mathbb{G}_m, 1) \wedge T) \cong \pi^{2,1}(\mathbb{A}^2 - \{0\}, (1, 1)) \cong \\ &\cong \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty(\mathbb{A}^2 - \{0\}, (1, 1)), \Sigma^\infty(\mathbb{P}^1, [1 : 1])) \end{aligned}$$

subject to the relation

$$\sigma^\pi \Sigma_T \Sigma^{1,1} \eta = \Sigma_T^\infty \vartheta H.$$

Definition 6. Let X be a smooth variety and $f \in \mathcal{O}_X^*$ be an invertible function on X . This function defines an automorphism

$$f_T: X_+ \wedge T \rightarrow X_+ \wedge T$$

via $f_T(x, t) = (x, f(x)t)$. We call the element

$$\langle f \rangle_\pi = \Sigma_T^{-1} f_T^\pi \Sigma_T 1 \in \pi^{0,0}(X)$$

the *symbol associated to f in π* .

Remark 1. By the very definition we have $\langle -1 \rangle_\pi = \epsilon$.

Consider the embedding $\{0\}_+ \rightarrow \mathbb{A}_+^1$ and the localization sequence

$$\pi^{0,0}(\mathbb{A}_+^1) \rightarrow \pi^{0,0}(\mathbb{G}_{m+}) \xrightarrow{\partial} \pi^{1,0}(T).$$

Our goal is to prove the following theorem.

Theorem 1. *Let $\partial: \pi^{0,0}(\mathbb{G}_{m+}) \rightarrow \pi^{1,0}(T)$ be the connecting homomorphism in the localization sequence for the embedding $\{0\} \subset \mathbb{A}^1$. Then for the coordinate function $t \in \mathcal{O}_{\mathbb{G}_m}^*$ we have*

$$\partial(\langle -t^{-1} \rangle_\pi) = \Sigma_T \eta.$$

In order to prove this theorem we need the next lemmas describing ∂ in a convenient way. Set $\delta = \Sigma^{-1,0}(\rho^\pi)^{-1} \partial$,

$$\begin{array}{ccc} \pi^{p,q}(\mathbb{G}_{m+}) & \xrightarrow{\partial} & \pi^{p+1,q}(T) \\ \delta \downarrow & & \simeq \uparrow \rho^\pi \\ \pi^{p,q}(\mathbb{G}_m) & \xrightarrow[\simeq]{\Sigma^{1,0}} & \pi^{p+1,q}(\mathbb{G}_m \wedge S_s^1). \end{array}$$

Lemma 1. *For the natural map $r: \mathbb{G}_{m+} \rightarrow \mathbb{G}_m$ we have $\delta r^\pi = id$.*

Proof. We can rewrite the statement of the lemma as $\partial r^\pi = \rho^\pi \Sigma^{1,0}$. Let $i: \mathbb{G}_m \rightarrow \mathbb{A}^1$ and $i_+: \mathbb{G}_{m+} \rightarrow \mathbb{A}_+^1$ be the natural embeddings and let $j_1: \mathbb{A}_+^1 \rightarrow \text{Cone}(i_+)$ and $j_2: \text{Cone}(i_+) \rightarrow \text{Cone}(j_1)$ be the natural maps for the cone construction.

Consider the following diagram.

$$\begin{array}{ccccc}
 \mathbb{G}_m \wedge S_s^1 & \xleftarrow[\simeq]{\rho_2} & Cone(i) & \xrightarrow[\simeq]{\rho_1} & T \\
 \uparrow r \wedge id & & \uparrow \simeq v & \nearrow \simeq \psi_1 & \\
 & & Cone(i_+) & & \\
 & \nwarrow u & \downarrow j_2 & & \\
 \mathbb{G}_{m+} \wedge S_s^1 & \xleftarrow[\simeq]{w} & Cone(j_1) & &
 \end{array}$$

Here ψ_1 and w are induced by $\Delta^1 \rightarrow pt$, v is induced by $\{+\} \rightarrow \{1\}$ and u is induced by $\mathbb{A}_+^1 \rightarrow pt$. One can check that this diagram is commutative in the homotopy category. By the very definition we have

$$\partial r^\pi = (wj_2\psi_1^{-1})^\pi \Sigma^{1,0} r^\pi = ((r \wedge id)wj_2\psi_1^{-1})^\pi \Sigma^{1,0},$$

thus it is sufficient to show

$$(r \wedge id)wj_2\psi_1^{-1} = \rho_2\rho_1^{-1} = \rho$$

and this follows from the commutativity of the above diagram. \square

Suspending δ with Σ_T and shifting the indices we obtain a homomorphism $\delta_T = \Sigma_T \delta \Sigma_T^{-1}$,

$$\begin{array}{ccc}
 \pi^{p,q}(\mathbb{G}_{m+} \wedge T) & \xrightarrow{\delta_T} & \pi^{p,q}(\mathbb{G}_m \wedge T) \\
 \Sigma_T \uparrow \simeq & & \Sigma_T \uparrow \simeq \\
 \pi^{p-2,q-1}(\mathbb{G}_{m+}) & \xrightarrow{\delta} & \pi^{p-2,q-1}(\mathbb{G}_m).
 \end{array}$$

Set $Y = (\mathbb{A}^2 - \{0\}) / ((\mathbb{A}^1 \times \mathbb{G}_m) \cup (\{1\} \times \mathbb{A}^1))$ and consider the following diagram

$$\begin{array}{ccccc}
 \pi^{p,q}((\mathbb{A}^2 - \{0\}) / (\mathbb{A}^1 \times \mathbb{G}_m)) & \xrightarrow[\simeq]{\phi^\pi} & \pi^{p,q}(\mathbb{G}_{m+} \wedge T) & & \\
 \psi_1^\pi \downarrow & \nwarrow \psi_2^\pi & \delta_T \downarrow \uparrow (r \wedge id)^\pi & & \\
 \pi^{p,q}(\mathbb{A}^2 - \{0\}, (1, 1)) & \xleftarrow[\simeq]{\sigma_1^\pi} \pi^{p,q}(Y) & \xrightarrow[\simeq]{\sigma_2^\pi} \pi^{p,q}(\mathbb{G}_m \wedge T) & &
 \end{array}$$

where ψ_1 and ψ_2 are induced by the identity map on $\mathbb{A}^2 - \{0\}$, ϕ^π is the excision isomorphism induced by the embedding $\mathbb{G}_m \times \mathbb{A}^1 \subset \mathbb{A}^2 - \{0\}$ and $r: \mathbb{G}_{m+} \rightarrow \mathbb{G}_m$ is the natural map.

Lemma 2. *In the above diagram we have*

$$\delta_T = \sigma_2^\pi (\sigma_1^\pi)^{-1} \psi_1^\pi (\phi^\pi)^{-1}.$$

Proof. Put $\delta'_T = \sigma_2^\pi (\sigma_1^\pi)^{-1} \psi_1^\pi (\phi^\pi)^{-1}$. By Lemma 1 we have

$$\delta_T (r \wedge id)^\pi = \Sigma_T \delta \Sigma_T^{-1} (r \wedge id)^\pi = \Sigma_T \delta r^\pi \Sigma_T^{-1} = id.$$

On the other hand, commutativity of the diagram

$$\begin{array}{ccccc}
 (\mathbb{A}^2 - \{0\})/(\mathbb{A}^1 \times \mathbb{G}_m) & \xleftarrow[\simeq]{\phi} & \mathbb{G}_{m+} \wedge T & & \\
 \psi_1 \uparrow & \searrow \psi_2 & \downarrow r \wedge id & & \\
 (\mathbb{A}^2 - \{0\}, (1, 1)) & \xrightarrow[\simeq]{\sigma_1} Y \xleftarrow[\simeq]{\sigma_2} & \mathbb{G}_m \wedge T & &
 \end{array}$$

implies $(r \wedge id)\phi^{-1}\psi_1\sigma_1^{-1}\sigma_2 = id$ and

$$\delta'_T(r \wedge id)^\pi = \sigma_2^\pi(\sigma_1^\pi)^{-1}\psi_1^\pi(\phi^\pi)^{-1}(r \wedge id)^\pi = id.$$

At last, set $s_1: \mathbb{G}_{m+} \wedge T \rightarrow T$ and $s_2: (\mathbb{A}^2 - \{0\})/(\mathbb{A}^1 \times \mathbb{G}_m) \rightarrow T$ for the natural projections $(x, y) \mapsto y$ and note that $s_2\phi = s_1$. Recall that the long exact sequence for the embedding $\{0\} \rightarrow \mathbb{A}^1$ admits a splitting yielding

$$\begin{aligned}
 \ker \delta_T &= \Sigma_T \ker \partial = \Sigma_T(\text{Im}(\pi^{p,q}(\mathbb{A}_+^1) \rightarrow \pi^{p,q}(\mathbb{G}_{m+}))) = \\
 &= \Sigma_T(\text{Im}(\pi^{p,q}(pt_+) \rightarrow \pi^{p,q}(\mathbb{G}_{m+}))) = s_1^\pi(\pi^{p,q}(T)).
 \end{aligned}$$

We have $s_2\psi_1 = 0$ in the homotopy category, so $\delta'_T\phi^\pi s_2^\pi = 0$. Thus we obtain

$$\ker \delta_T = s_1^\pi(\pi^{p,q}(T)) = \phi^\pi s_2^\pi(\pi^{p,q}(T)) \subset \ker \delta'_T.$$

To sum up, there are two homomorphisms δ_T, δ'_T with the same splitting and $\ker \delta_T \subset \ker \delta'_T$. Then for every $a \in \pi^{p,q}(\mathbb{G}_{m+} \wedge T)$ we have

$$\delta_T(a - (r \wedge id)^\pi \delta_T(a)) = 0 = \delta'_T(a - (r \wedge id)^\pi \delta_T(a)) = \delta'_T(a) - \delta_T(a),$$

yielding $\delta_T = \delta'_T$. \square

Proof of Theorem 1. Consider the morphism $m: \mathbb{G}_{m+} \wedge T \rightarrow \mathbb{G}_{m+} \wedge T$ given by $m(t, x) = (t, -x/t)$. Unraveling the definitions, we need to show

$$\partial \Sigma_T^{-1} m^\pi \Sigma_T(1) = \Sigma_T \Sigma^{-1, -1} \Sigma_T^{-1} (\sigma^\pi)^{-1} (\Sigma^\infty \vartheta H).$$

The element $m^\pi \Sigma_T 1 \in \pi^{2,1}(\mathbb{G}_{m+} \wedge T)$ could be represented by the Σ_T^∞ -suspension of the composition

$$\mathbb{G}_{m+} \wedge T \xrightarrow{\tilde{H}_1} T \xrightarrow{\rho} S^{2,1},$$

with \tilde{H}_1 given by $\tilde{H}_1(t, x) = -x/t$. We can rewrite this composition as $\rho \tilde{H}_1 = \rho \vartheta_2^{-1} H_1$ with $H_1: \mathbb{G}_{m+} \wedge T \rightarrow \mathbb{P}^1/\mathbb{A}^1$ given by $H_1(t, x) = [x : -t]$. Hence on the left-hand side we have $\partial \Sigma_T^{-1} (\Sigma_T^\infty \rho \vartheta_2^{-1} H_1)$.

Unraveling the definition of δ_T , by Lemma 2 we obtain

$$\Sigma_T \Sigma^{-1,0} (\rho^\pi)^{-1} \partial \Sigma_T^{-1} = (\sigma^\pi)^{-1} \psi_1^\pi (\phi^\pi)^{-1},$$

$$\partial \Sigma_T^{-1} = \rho^\pi \Sigma^{1,0} \Sigma_T^{-1} (\sigma^\pi)^{-1} \psi_1^\pi (\phi^\pi)^{-1}.$$

Thus on the left-hand side we have

$$\rho^\pi \Sigma^{1,0} \Sigma_T^{-1} (\sigma^\pi)^{-1} \psi_1^\pi (\phi^\pi)^{-1} (\Sigma^\infty \rho \vartheta_2^{-1} H_1).$$

Consider the following commutative diagram.

$$\begin{array}{ccccc}
 \mathbb{P}^1 & \xrightarrow[\simeq]{\vartheta_1} & \mathbb{P}^1/\mathbb{A}^1 & \xrightarrow[\tau]{=} & \mathbb{P}^1/\mathbb{A}^1 \\
 \uparrow H & \nearrow H_3 & & \nearrow H_2 & \uparrow H_1 \\
 (\mathbb{A}^2 - \{0\}, (1, 1)) & \xrightarrow{\psi_1} & (\mathbb{A}^2 - \{0\})/(\mathbb{A}^1 \times \mathbb{G}_m) & \xleftarrow[\simeq]{\phi} & \mathbb{G}_{m+} \wedge T
 \end{array}$$

Here H_1 and H_2 are given by $(x, y) \mapsto [y : -x]$, H and H_3 are given by $(x, y) \mapsto [x : y]$, $\mathbb{P}^1 \cong \mathbb{P}^1/\mathbb{A}^1$ is the canonical isomorphism and τ is given by $[x, y] \mapsto [y, -x]$. Recall that in the homotopy category $\tau = id$, thus we have

$$\psi_1^\pi(\phi^\pi)^{-1}(\Sigma_T^\infty \rho \vartheta_2^{-1} H_1) = \Sigma_T^\infty (\rho \vartheta_2^{-1} H_1 \phi^{-1} \psi_1) = \Sigma_T^\infty \rho \vartheta_2^{-1} \vartheta_1 H = \Sigma_T^\infty \vartheta H.$$

Summing up the above considerations, it is left to show that

$$\rho^\pi \Sigma^{1,0} \Sigma_T^{-1} (\sigma^\pi)^{-1} (\Sigma_T^\infty \vartheta H) = \Sigma_T \Sigma^{-1,-1} \Sigma_T^{-1} (\sigma^\pi)^{-1} (\Sigma_T^\infty \vartheta H),$$

and it follows from the definition of Σ_T . □

4. SPECIAL LINEAR AND SYMPLECTIC ORIENTATIONS.

In this section we briefly recall the definitions of different types of orientations, a detailed exposition could be found in [PW10a, PW10b]. For the sake of uniformity mostly the case of a representable ring cohomology theory is treated. Fix for this section a commutative monoid $A \in \mathcal{SH}(k)$ representing a ring cohomology theory $A^{*,*}(-)$.

Roughly speaking, an orientation on a cohomology theory is a rule that fixes for every vector bundle E over every smooth variety X a Thom class $th(E) \in A^{*,*}(Th(E))$ such that

$$- \cup th(E): A^{*,*}(X) \rightarrow A^{*+2\text{rank } E, *+\text{rank } E}(Th(E))$$

is an isomorphism and these classes satisfy certain natural properties [PS03]. Particular types of orientation fix such classes only for the vector bundles with additional structure, for example for the symplectic or for the special linear ones. Note that these classes usually do depend on the additional structure, i.e. for the same vector bundle with different symplectic forms one could have different Thom classes.

Definition 7. A *special linear bundle* over a variety X is a pair (E, λ) with $E \rightarrow X$ a vector bundle and $\lambda: \det E \xrightarrow{\sim} \mathcal{O}_X$ an isomorphism of line bundles.

Definition 8. A (*normalized*) *special linear orientation* [PW10c, Definition 5.1] on a ring cohomology theory $A^{*,*}(-)$ is a rule which assigns to every special linear bundle (E, λ) of rank n over a smooth variety X a functorial and multiplicative class $th(E, \lambda) \in A^{2n,n}(Th(E))$ such that the maps

$$- \cup th(E, \lambda): A^{*,*}(X) \rightarrow A^{*+2n, *+n}(Th(E))$$

are isomorphisms and for the zero bundle $\mathbf{0} \rightarrow pt$ and for the trivial special linear line bundle L over a point we have $th(\mathbf{0}) = 1 \in A^{0,0}(pt)$ and $th(L) =$

$\Sigma_T 1 \in A^{2,1}(T)$ respectively. The class $th(E, \lambda)$ is a *Thom class* of the special linear bundle and

$$e(E, \lambda) = z^A th(E, \lambda) \in A^{2n,n}(X)$$

with natural $z: X \rightarrow Th(E)$ is its *Euler class*. We call a ring cohomology theory with a normalized special linear orientation an *SL-oriented cohomology theory*.

One can give an analogous definition for the symplectic orientation on a cohomology theory.

Definition 9. A *(normalized) symplectic orientation* [PW10a, Definition 14.2] on a ring cohomology theory $A^{*,*}(-)$ is a rule which assigns to every symplectic bundle (E, ϕ) of rank n over a smooth variety X a functorial and multiplicative class $th(E, \phi) \in A^{2n,n}(Th(E))$ such that the maps

$$- \cup th(E, \phi): A^{*,*}(X) \rightarrow A^{*+2n, *+n}(Th(E))$$

are isomorphisms and for the zero bundle $\mathbf{0} \rightarrow pt$ and for the hyperbolic bundle (H, ϕ) of rank two over a point we have $th(\mathbf{0}) = 1 \in A^{0,0}(pt)$ and $th(H, \phi) = \Sigma_T^2 1 \in A^{4,2}(T \wedge T)$ respectively. The class $th(E, \phi)$ is a *Thom class* of the symplectic bundle.

Remark 2. One can introduce analogous notions of orientations on a ring cohomology theory $A(-)$ defined on the category of pairs (X, U) with a smooth variety X and open subvariety U repeating the above definitions and writing $A(E, E - X)$ for $A(Th(E))$, details could be found in [PS03]. Note that one does not have canonical suspension elements in $A(\mathbb{A}^n, \mathbb{A}^n - \{0\})$ analogous to $\Sigma_T^n 1$, so the normalization property involving $\Sigma_T 1$ from Definition 8 and the corresponding property for the normalized symplectic orientation involving $\Sigma_T^2 1$ should be dropped.

Definition 10. Let $A(-)$ be an *SL-oriented* cohomology theory, X be a smooth variety and $f \in \mathcal{O}_X^*$ be an invertible function. Regarding f as an isomorphism $f_T: \mathcal{O}_X \rightarrow \mathcal{O}_X$ given by $f_T(x, t) = (x, f(x)t)$ one obtains classes $th(\mathcal{O}_X, f_T)$. A *symbol* $\langle f \rangle_A$ is the unique element of $A(X)$ subject to the relation

$$\langle f \rangle_A \cup th(\mathcal{O}_X, 1) = th(\mathcal{O}_X, f_T).$$

For a representable cohomology theory $A^{*,*}(-)$ with a normalized special linear orientation these symbols agree with the ones arising from $\langle f \rangle_\pi$ and the monoid structure morphism $e_A: \mathbb{S} \rightarrow A$.

Remark 3. The above definition is consistent for every cohomology theory possessing Thom classes for the trivialized linear bundles, i.e. $A(pt)$ -module isomorphisms $A(pt) \cong A(\mathbb{A}^1, \mathbb{G}_m)$.

Every symplectic bundle is a special linear bundle in a natural way, so a special linear orientation on a ring cohomology theory induces a symplectic one, and a general orientation induces a special linear one. Thus one has a variety of examples of *SL-oriented* cohomology theories arising from the oriented ones. Typical examples of *SL-oriented* but not oriented cohomology theories are hermitian K -theory introduced by Schlichting [Sch10a, Sch12] and derived Witt

groups defined by Balmer [Bal99, Bal05], we recall the constructions in the next section. Further examples are given by the algebraic cobordism MSL and MSp [PW10c]. We briefly recall some definitions related to the spectrum MSL .

Definition 11. Let $E(n, m)$ be the tautological vector bundle of rank n over $Gr(n, m)$. The special linear Grassmannian is the complement to the zero section of $\det E(n, m)$,

$$SGr(n, m) = (\det E(n, m))^0.$$

Denote by $\mathcal{T}(n, m)$ the tautological special linear bundle of rank n over $SGr(n, m)$ and set $Th(n, m) = Th(\mathcal{T}(n, m))$ for its Thom space. For the natural monomorphisms $Th(n, m) \rightarrow Th(n, m+1)$ set

$$MSL_n = \varinjlim_{m \in \mathbb{N}} Th(n, m).$$

In order to write down the bonding maps and the monoid structure it is more convenient to work in the category of symmetric T^2 -spectra, see [PW10c] for the details. Sometimes one prefers to work with the spectrum MSL^{fin} with $MSL_n^{fin} = Th(n, n^2)$. The natural map $MSL^{fin} \rightarrow MSL$ induces an isomorphism in $\mathcal{SH}(k)$.

The cobordism cohomology theories are the universal ones in the sense of the following theorems (see [PW10c, Theorems 12.2, 13.2, 5.9]).

Theorem 2. Suppose A is a commutative monoid in $\mathcal{SH}(k)$ with a normalized symplectic orientation on $A^{*,*}(-)$ given by Thom classes $th^A(E, \theta)$. Then there exists a unique morphism $\varphi^{Sp}: MSp \rightarrow A$ in $\mathcal{SH}(k)$ such that for every symplectic bundle (E, θ) over every smooth variety X one has

$$\varphi_{Th(E)}^{Sp}(th^{MSp}(E, \theta)) = th^A(E, \theta).$$

Moreover, this morphism is a homomorphism of commutative monoids.

Theorem 3. Suppose A is a commutative monoid in $\mathcal{SH}(k)$ with a normalized special linear orientation on $A^{*,*}(-)$ given by Thom classes $th^A(E, \lambda)$. Then there exists a morphism $\varphi^{SL}: MSL \rightarrow A$ in $\mathcal{SH}(k)$ such that

$$\varphi_{Th(E)}^{SL}(th^{MSL}(E, \lambda)) = th^A(E, \lambda)$$

for every special linear bundle (E, λ) . It satisfies $\varphi_{pt}(1_{MSL}) = 1_A$.

There is a \lim^1 -obstruction for the morphism from Theorem 3 to be a homomorphism of monoids. As we will see shortly, this obstruction makes no matter for small pointed motivic spaces.

Definition 12. A pointed motivic space Y is called *small* if $Hom_{\mathcal{SH}(k)}(\Sigma^\infty Y, -)$ commutes with arbitrary coproducts.

Lemma 3. For a morphism φ^{SL} from Theorem 3, small pointed motivic spaces Y, Y' and arbitrary elements $\alpha \in MSL^{*,*}(Y), \alpha' \in MSL^{*,*}(Y')$ one has

$$\varphi_{Y \wedge Y'}^{SL}(\alpha \times \alpha') = \varphi_Y^{SL}(\alpha) \times \varphi_{Y'}^{SL}(\alpha').$$

In particular, φ_Y^{SL} is a homomorphism of $\pi^{*,*}(pt)$ -algebras.

Proof. Suspending one may pass to

$$\Sigma^{p,q}\alpha \in MSL^{2n,n}(Y \wedge S^{p,q}), \quad \Sigma^{p',q'}\alpha' \in MSL^{2n',n'}(Y' \wedge S^{p',q'}).$$

Since $MSL \cong MSL^{fin}$ in $\mathcal{SH}(k)$ and Y is small, there are canonical isomorphisms (for the second one see [Voe98, Theorem 5.2])

$$\begin{aligned} MSL^{2n,n}(Y \wedge S^{p,q}) &\cong \text{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty(Y \wedge S^{p,q})[-n], MSL^{fin}) \cong \\ &\cong \varinjlim_{i \in \mathbb{N}} \text{Hom}_{H_\bullet(k)}(Y \wedge S^{p,q} \wedge T^{\wedge i}, Th(n+i, (n+i)^2)). \end{aligned}$$

These isomorphisms imply that there exists some $i \in \mathbb{N}$ and

$$f \in \text{Hom}_{H_\bullet(k)}(Y \wedge S^{p,q} \wedge T^{\wedge i}, Th(n+i, (n+i)^2))$$

such that

$$\Sigma_T^i \Sigma^{p,q} \alpha = f^{MSL} th^{MSL}(\mathcal{T}(n+i, (n+i)^2)).$$

By a similar argument we obtain that

$$\Sigma_T^{i'} \Sigma^{p',q'} \alpha' = f'^{MSL} th^{MSL}(\mathcal{T}(n'+i', (n'+i')^2))$$

for some $i', p', q' \in \mathbb{N}$ and $f \in \text{Hom}_{H_\bullet(k)}(Y' \wedge S^{p',q'} \wedge T^{\wedge i'}, Th(n'+i', (n'+i')^2))$. Put $m = n+i, m' = n'+i'$ and consider the following diagram.

$$\begin{array}{ccc} \Sigma_T^\infty MSL_m^{fin}[-m] \wedge \Sigma_T^\infty MSL_{m'}^{fin}[-m'] & \xrightarrow{th_m^{BO} \wedge th_{m'}^{BO}} & BO \wedge BO \\ \downarrow \mu_{m,m'} & \searrow th_m^{MSL} \wedge th_{m'}^{MSL} & \nearrow \varphi^{SL} \wedge \varphi^{SL} \\ & MSL \wedge MSL & \\ & \downarrow \mu_{MSL} & \downarrow \mu_{BO} \\ \Sigma_T^\infty MSL_{m+m'}^{fin}[-m-m'] & \xrightarrow{th_{m+m'}^{BO}} & BO \\ & \searrow th_{m+m'}^{MSL} & \nearrow \varphi^{SL} \\ & MSL & \end{array}$$

Here $\mu_{m,m'}$ is induced by the canonical embedding

$$\mathcal{T}(m, m^2) \times \mathcal{T}(m', m'^2) \rightarrow \mathcal{T}(m+m', (m+m')^2)$$

and all the maps th are given by the corresponding Thom classes, for example th_m^{BO} is induced by

$$th^{BO}(\mathcal{T}(m, m^2)) \in BO^{2m,m}(Th(m, m^2)) \cong \text{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty MSL_m^{fin}[-m], BO).$$

The left and the back squares commute by the multiplicativity of the Thom classes. The triangles commute by the choice of φ^{SL} . Hence

$$\begin{aligned} \mu_{BO}(\varphi^{SL} \wedge \varphi^{SL})(th_m^{MSL} \wedge th_{m'}^{MSL})(\Sigma_T^\infty f[-m] \wedge \Sigma_T^\infty f'[-m']) &= \\ = \varphi^{SL} \mu_{MSL}(th_m^{MSL} \wedge th_{m'}^{MSL})(\Sigma_T^\infty f[-m] \wedge \Sigma_T^\infty f'[-m']), \end{aligned}$$

and, omitting indices at φ^{SL} ,

$$\varphi^{SL}(\Sigma_T^i \Sigma^{p,q} \alpha \times \Sigma_T^{i'} \Sigma^{p',q'} \alpha') = \varphi^{SL}(\Sigma_T^i \Sigma^{p,q} \alpha) \times \varphi^{SL}(\Sigma_T^{i'} \Sigma^{p',q'} \alpha').$$

The morphisms φ_Y^{SL} and $\varphi_{Y'}^{SL}$ are agreed with suspension isomorphisms, so the claim follows from the above equality. \square

5. SCHLICHTING'S HERMITIAN K -THEORY AND WITT GROUPS.

In this section we recall the definitions of hermitian K -theory introduced by Schlichting [Sch10a] and the Gille-Nenashev pairing [GN03]. In our exposition we mainly follow [PW10b].

For a smooth variety X consider the category $Vect(X)$ of vector bundles over X and let $Ch^b(Vect(X))$ be the additive category of bounded complexes of vector bundles on X . We consider $Ch^b(Vect(X))$ as a complicial exact category with the weak equivalences ω_X being quasi-isomorphisms and the conflations being the degreewise-split short exact sequences. We endow $Ch^b(Vect(X))$ with the duality consisting of the functor $(-)^{\vee} = Hom_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and the natural biduality map $\eta_X: 1 \xrightarrow{\sim} (-)^{\vee\vee}$.

Now consider an open subvariety $U \subset X$ and set $Z = X - U$. Let $Ch_Z^b(Vect(X))$ be the full additive subcategory of complexes which are acyclic on U . We can regard $Ch_Z^b(Vect(X))$ as a complicial exact category with the same weak equivalences, conflations and duality as $Ch^b(Vect(X))$.

In [Sch10b] Schlichting defines the hermitian K -theory space of a complicial exact category with weak equivalences and duality in the style of Waldhausen's K -theory. For the considered categories of bounded complexes we denote the corresponding spaces by $KO(X)$ and $KO(X, U)$. More generally we write $KO^{[n]}(X)$ and $KO^{[n]}(X, U)$ for the hermitian K -theory spaces for the n -th shifted duality and $KO_i^{[n]}(X)$ and $KO_i^{[n]}(X, U)$ for its homotopy groups.

There is a fibration sequence up to homotopy

$$KO^{[n]}(X, U) \rightarrow KO^{[n]}(X) \rightarrow KO^{[n]}(U)$$

and therefore long exact sequences

$$\dots \rightarrow KO_i^{[n]}(X, U) \rightarrow KO_i^{[n]}(X) \rightarrow KO_i^{[n]}(U) \xrightarrow{\partial} KO_{i-1}^{[n]}(X, U) \rightarrow \dots$$

We don't have the negative groups $KO_i^{[n]}(X)$ and $KO_i^{[n]}(X, U)$ for $i < 0$, so the above sequence terminates at $i = 0$. It is well known that appropriate Witt groups can substitute the negative groups in hermitian K -theory, so we recall the definitions [Bal99, Wal03].

Let $D^b(Vect(X))$ and $D_Z^b(Vect(X))$ be the derived categories of the exact categories $Ch^b(Vect(X))$ and $Ch_Z^b(Vect(X))$ equipped with the untwisted duality consisting of the functor $(-)^{\vee} = Hom_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and the natural biduality map $\eta_X: 1 \xrightarrow{\sim} (-)^{\vee\vee}$ as above. We denote by $D^b(Vect(X))[n]$ and $D_Z^b(Vect(X))[n]$ the same derived categories but with n -th shifted dualities. All the considered derived categories contain $\frac{1}{2}$, so we omit this assumption in the next definitions.

A symmetric object for a derived category C with a duality is a pair (A, α) of $A \in Ob(C)$ and a morphism $\alpha: A \rightarrow A^{\vee}$ agreed with the duality. A symmetric space (A, α) is a symmetric object with α being an isomorphism. There are

obvious notions of the isomorphism of symmetric spaces and of the orthogonal sum $(A, \alpha) \perp (B, \beta)$ of symmetric spaces. For every symmetric object (A, α) there exists a canonical symmetric space $Cone(A, \alpha)$ for the 1-st shifted duality, in particular, for every object $A \in Ob(C)$ there is a hyperbolic symmetric space $H(A) = Cone(A, 0)$. The Grothendieck-Witt group $GW(C)$ of a small triangulated category with duality is the quotient of the free abelian group on the isomorphism classes of symmetric spaces by the relations $[(A, \alpha) \perp (B, \beta)] = [(A, \alpha)] + [(B, \beta)]$ and $Cone(A, \alpha) = H(A)$, where for the second class of relations we regard A as an object of the derived category with (-1) -st shifted duality. The Witt group $W(C)$ is the quotient of the Grothendieck-Witt group by the subgroup generated by the classes of the hyperbolic spaces.

We write

$$GW^n(X) = GW(D^b(Vect(X))[n]), \quad GW^n(X, U) = GW(D_Z^b(Vect(X))[n]),$$

$$W^n(X) = W(D^b(Vect(X))[n]), \quad W^n(X, U) = W(D_Z^b(Vect(X))[n])$$

for the corresponding Grothendieck-Witt and Witt groups arising from the derived category of the vector bundles over a smooth variety X . There is a long exact sequence [Wal03, Theorem 2.4]

$$\begin{aligned} GW^n(X, U) &\rightarrow GW^n(X) \rightarrow GW^n(U) \xrightarrow{\partial} \\ &\rightarrow W^{n+1}(X, U) \rightarrow W^{n+1}(X) \rightarrow W^{n+1}(U) \xrightarrow{\partial} \\ &\rightarrow W^{n+2}(X, U) \rightarrow W^{n+2}(X) \rightarrow W^{n+2}(U) \xrightarrow{\partial} \dots \end{aligned}$$

Recall that there are natural identifications

$$KO_0^{[n]}(X) \cong GW^n(X), \quad KO_0^{[n]}(X, U) \cong GW^n(X, U),$$

so we set

$$KO_i^{[n]}(X) = W^{n-i}(X), \quad KO_i^{[n]}(X, U) = W^{n-i}(X, U),$$

for $i < 0$. Then the long exact sequences for $KO_i^{[n]}$ and W^n could be combined into one long exact sequence for the hermitian K -theory.

For a smooth variety X and $s \in \mathcal{O}_X^*$ we write $\langle s \rangle$ for the class

$$\langle s \rangle = [(\mathcal{O}_X, s)] \in GW^0(X) \cong KO_0^{[0]}(X).$$

Gille and Nenashev [GN03] have introduced pairings for the Witt groups of triangulated categories. Their construction could be adapted to hermitian K -theory to give pairings on KO , see [PW10b],

$$\begin{aligned} \star: KO_i^{[n]}(X, U) \times KO_0^{[m]}(X, V) &\rightarrow KO_i^{[n+m]}(X, U \cup V), \\ \star: KO_0^{[n]}(X, U) \times KO_i^{[m]}(X, V) &\rightarrow KO_i^{[n+m]}(X, U \cup V). \end{aligned}$$

The left and right pairings constructed by Gille and Nenashev coincide up to $(-1)^{nm}$, so in the setting of $KO_i^{[n]}$ the definition of the left pairing is corrected by $\langle -1 \rangle^{nm}$ in order to obtain the agreed pairings for $i = 0$. Restricting this pairings to $i = 0$ we obtain the multiplication

$$\star: GW^n(X, U) \times GW^m(X, V) \rightarrow GW^{n+m}(X, U \cup V).$$

Recall that for $[(A_\bullet, \alpha)] \in GW^n(X, U)$, $[(B_\bullet, \beta)] \in GW^m(X, V)$ its product is given by

$$[(A_\bullet, \alpha)] \star [(B_\bullet, \beta)] = [(A_\bullet \otimes B_\bullet, \alpha \star \beta)] \in GW^{n+m}(X, U \cup V)$$

where $\alpha \star \beta$ equals to $\alpha \otimes \beta$ up to some signs and identification

$$A_\bullet^\vee[n] \otimes B_\bullet^\vee[m] = (A_\bullet \otimes B_\bullet)^\vee[n+m].$$

Note that we have $KO_i^{[n]}(X, U) = W^{n-i}(X, U)$ for $i < 0$, so the right Gille-Nenashev pairing gives a product

$$\star: \bigoplus_{n \in \mathbb{Z}, i < 0} KO_i^{[n]}(X, U) \times \bigoplus_{n \in \mathbb{Z}, i < 0} KO_i^{[n]}(X, V) \rightarrow \bigoplus_{n \in \mathbb{Z}, i < 0} KO_i^{[n]}(X, U \cup V).$$

For a special linear bundle (E, λ) of rank n over a smooth variety X one can construct a Thom class for hermitian K -theory using the method introduced by Nenashev for Witt groups [Ne07]. Let $p: E \rightarrow X$ be the structure map. Consider the pullback $p^*E = E \oplus E \rightarrow E$. There is a canonical diagonal section $s: E \rightarrow E \oplus E$ that defines a map $p^*E^\vee \rightarrow \mathcal{O}_E$ via the pairing $p^*E \otimes p^*E^\vee \rightarrow \mathcal{O}_E$. This map give rise to the Koszul complex

$$K(E) = (0 \rightarrow \Lambda^n p^*E^\vee \rightarrow \Lambda^{n-1} p^*E^\vee \rightarrow \dots \rightarrow \Lambda^2 p^*E^\vee \rightarrow p^*E^\vee \rightarrow \mathcal{O}_E \rightarrow 0)$$

which is treated as a chain complex located in homological degrees n through 0 . It is well known that this complex is exact off X . There is a canonical isomorphism $\Theta(E): K(E) \xrightarrow{\sim} K(E)^\vee[n] \otimes \det p^*E^\vee$, thus λ induces an isomorphism $\Theta(E, \lambda): K(E) \xrightarrow{\sim} K(E)^\vee[n]$. One can show that $(K(E), \Theta(E, \lambda))$ is a symmetric space for the the n -th shifted duality. The corresponding element in the Grothendieck-Witt group

$$th^{KO}(E, \lambda) = [K(E), \Theta(E, \lambda)] \in GW^n(E, E - X) = KO_0^{[n]}(E, E - X)$$

represents the Thom class of the special linear bundle (E, λ) for the hermitian K -theory, in particular, we have the isomorphisms [PW10b, Theorem 5.1]

$$- \cup th^{KO}(E, \lambda): KO_i^{[m]}(X) \xrightarrow{\sim} KO_i^{[m+n]}(E, E - X).$$

We finish this section with the following lemmas relating the symbols from Definition 6 in the case of hermitian K -theory to quadratic forms.

Lemma 4. *Let X be a smooth variety and $f \in \mathcal{O}_X^*$. Then*

$$\langle f \rangle_{KO} = \langle f \rangle.$$

Proof. Let $f_T: \mathcal{O}_X \rightarrow \mathcal{O}_X$ be the isomorphism given by $f_T(x, t) = (x, f(x)t)$. The equality

$$\langle f \rangle \star th^{KO}(\mathcal{O}_X, 1) = th^{KO}(\mathcal{O}_X, f_T)$$

follows from the definitions of the Thom class and the pairing. On the other hand, the symbol $\langle f \rangle_{KO}$ is defined by means of the similar equality

$$\langle f \rangle_{KO} \cup th^{KO}(\mathcal{O}_X, 1) = th^{KO}(\mathcal{O}_X, f_T),$$

so the claim follows. \square

Lemma 5. *Let $\partial: W^0(\mathbb{G}_m) \rightarrow W^1(\mathbb{A}^1, \mathbb{G}_m)$ be the connecting homomorphism in the localization sequence for the inclusion $\{0\} \rightarrow \mathbb{A}^1$ and let $t \in \mathcal{O}_{\mathbb{G}_m}^*$ be the coordinate function. Then one has*

$$\partial(\langle t \rangle) = th(\mathcal{O}_{pt}, 1).$$

Proof. In order to compute $\partial(\langle t \rangle)$ one should write down the cone for the symmetric space (A_\bullet, t) with A_\bullet being the complex concentrated in the zeroth degree and $A_0 = \mathcal{O}_{\mathbb{A}^1}$. It turns out that this cone coincides with the desired Thom class, $[(\mathcal{O}_{\mathbb{A}^1} \xrightarrow{t} \mathcal{O}_{\mathbb{A}^1}, \Theta(\mathcal{O}_{\mathbb{A}^1}, 1))]$. \square

6. THE T -SPECTRUM BO AND THE COHOMOLOGY THEORY $BO^{*,*}$.

In [PW10b] Panin and Walter constructed a commutative monoid (BO, m, e) in $\mathcal{SH}(k)$ representing hermitian K -theory in the following precise sense (see [PW10b, Corollary 7.3, Theorem 13.4] and [PW10d, Lemma 4.4]):

Theorem 4. *For every smooth variety X and open subset U there exist canonical functorial isomorphisms $\gamma: BO^{p,q}(X_+/U_+) \xrightarrow{\cong} KO_{2q-p}^{[q]}(X, U)$.*

- (1) *These isomorphisms agree with connecting homomorphisms ∂ in localization sequences.*
- (2) *The \cup -product on $BO^{2*,*}(-)$ arising from the monoid structure of BO is agreed with the Gille-Nenashev right pairing on $KO_0^{[*]}(-)$.*
- (3) *There are canonical Thom classes $th^{BO}(E, \lambda)$ defined for special linear bundles such that $\gamma(th^{BO}(E, \lambda)) = th^{KO}(E, \lambda)$ and $th^{BO}(\mathcal{O}_X, 1) = \Sigma_T 1$.*
- (4) *$\gamma(1) = \langle 1 \rangle$ and $\gamma(\epsilon) = \langle -1 \rangle$.*

Corollary 1. *For a smooth variety X and $f \in \mathcal{O}_X^*$ we have*

$$\gamma(\langle f \rangle_{BO}) = \langle f \rangle.$$

Proof. The symbols are defined via the cup-product with the Thom classes, so it follows from the theorem that $\gamma(\langle f \rangle_{BO}) = \langle f \rangle_{KO}$ and Lemma 4 gives the claim. \square

The above theorem provides natural isomorphisms $BO^{p,q}(X) \cong KO_{2q-p}^{[q]}(X) = W^{p-q}(X)$ for $2q - p < 0$. Recall that one has Gille-Nenashev pairing on the Witt groups. This pairing is compatible with the \cup -product on $BO^{*,*}$ in the sense of the following lemma.

Lemma 6. *For a smooth variety X and $a \in BO^{p,q}(X), b \in BO^{r,s}(X)$ with $2q - p, 2s - r \leq 0$ we have*

$$\gamma(a \cup b) = (-1)^{p(r-s)} \gamma(a) \star \gamma(b).$$

Proof. Proceed by induction on $p - 2q, r - 2s$. In the case $p - 2q = r - 2s = 0$ the claim follows from the item 2 of the above theorem. Suppose that $r - 2s > 0$ and consider the connecting homomorphisms in the long exact sequences for

the zero section of the trivial bundle $X \times \mathbb{A}^1 \rightarrow X$,

$$\begin{array}{ccccc} BO^{r+1,s+1}(X \times \mathbb{G}_m) & \xrightarrow{\partial_{BO}} & BO^{r+2,s+1}(X_+ \wedge T) & \xleftarrow[\simeq]{Th_{BO}} & BO^{r,s}(X) \\ \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\ KO_{2s-r+1}^{[s+1]}(X \times \mathbb{G}_m) & \xrightarrow{\partial_{KO}} & KO_{2s-r}^{[s+1]}(X \times \mathbb{A}^1, X \times \mathbb{G}_m) & \xleftarrow[\simeq]{Th_{KO}} & KO_{2s-r}^{[s]}(X) \end{array}$$

Here Th_{BO} and Th_{KO} are the Thom isomorphisms given by

$$Th_{BO}(x) = x \cup th^{BO}(\mathcal{O}_X, 1), \quad Th_{KO}(x) = x \star th^{KO}(\mathcal{O}_X, 1).$$

Consider the compositions $\partial'_{BO} = Th_{BO}^{-1} \partial_{BO}$, $\partial'_{KO} = Th_{KO}^{-1} \partial_{KO}$. A trivial line bundle possesses a section thus ∂_{BO} and ∂'_{BO} are surjective. Take

$$\bar{b} \in BO^{r+1,s+1}(X \times \mathbb{G}_m)$$

such that $\partial'_{BO}(\bar{b}) = b$ and proceed by induction using that ∂'_{BO} and ∂'_{KO} are homomorphisms of left $BO^{*,*}(X)$ and $KO_*^{[*]}(X)$ -modules respectively:

$$\begin{aligned} \gamma(a \cup b) &= \gamma(a \cup \partial'_{BO}(\bar{b})) = \gamma(\partial'_{BO}(a \cup \bar{b})) = \partial'_{KO}(\gamma(a \cup \bar{b})) = \\ &= (-1)^{p(r-s)} \partial'_{KO}(\gamma(a) \star \gamma(\bar{b})) = (-1)^{p(r-s)} \gamma(a) \star \partial'_{KO}(\gamma(\bar{b})) = \\ &= (-1)^{p(r-s)} \gamma(a) \star \gamma(\partial'_{BO}(\bar{b})) = (-1)^{p(r-s)} \gamma(a) \star \gamma(b). \end{aligned}$$

In case of $r - 2s = 0$ and $p - 2q > 0$ one should swap a and b applying graded commutativity, then compute using the above case and swap again:

$$\begin{aligned} \gamma(a \cup b) &= \gamma((-1)^{pr} \epsilon^{qs} \cup b \cup a) = (-1)^{pr+qs} (-1)^{r(p-q)} \gamma(b) \star \gamma(a) = \\ &= (-1)^{pr+qs+r(p-q)} (-1)^{(p-q)(r-s)} \gamma(a) \star \gamma(b) = (-1)^{p(r-s)} \gamma(a) \star \gamma(b). \quad \square \end{aligned}$$

Remark 4. Using the same arguments and the localization sequence of a triple one can show that analogous claim holds for $a \in BO^{p,q}(X, U)$, $b \in BO^{r,s}(X, V)$.

We have $BO^{-1,-1}(pt) \cong KO_{-1}^{[-1]}(pt) = W^0(pt)$, in particular, there is a distinguished element $-1 = \langle -1 \rangle \in W^0(pt)$. The next lemma presents the corresponding element in $BO^{-1,-1}(pt)$ arising from the stable cohomotopy groups.

Lemma 7. *For the stable Hopf map $\eta \in BO^{-1,-1}(pt)$ we have $\gamma(\eta) = -1$.*

Proof. By Theorem 1 we have $\partial(\langle -t^{-1} \rangle_{BO}) = \Sigma_T \eta$, thus

$$\gamma(\eta) \star th(\mathcal{O}_{pt}, 1) = \gamma(\Sigma_T \eta) = \gamma \partial(\langle -t^{-1} \rangle_{BO}) = \partial \gamma(\langle -t^{-1} \rangle_{BO}) = \partial(\langle -t^{-1} \rangle)$$

with the last one equality arising from Corollary 1. Multiplying by t^2 and applying Lemma 5 we obtain

$$\gamma(\eta) \star th(\mathcal{O}_{pt}, 1) = \partial(\langle -t^{-1} \rangle) = \partial(\langle -t \rangle) = -th(\mathcal{O}_{pt}, 1).$$

The claim follows since $-\star th(\mathcal{O}_{pt}, 1)$ is an isomorphism \square

The main result of this section states that inverting the stable Hopf map in $BO^{*,*}(X)$ one obtains the $(-1, -1)$ -commutative Laurent polynomial ring over $W^*(X)$ in η . The more precise statement follows.

Definition 13. Let X be a smooth variety and let

$$W^{*,*}(X) = W^*(X)[\eta, \eta^{-1}]$$

be a $(-1, -1)$ -commutative bigraded algebra with $\deg \eta = (-1, -1)$ and $\deg a = (2n, n)$ for $a \in W^n(X)$, i.e. for $a \in W^i(X), b \in W^j(X)$ one has

$$(a \star \eta^p) \star (b \star \eta^q) = (-1)^{(i+p)(j+q)+pq} (b \star \eta^q) \star (a \star \eta^p).$$

Define $\psi: BO^{*,*}(X) \rightarrow W^{*,*}(X)$ in the following way. For $a \in BO^{p,q}(X)$ set

$$\psi(a) = \begin{cases} \gamma(a) \star \eta^{p-2q}, & 2q - p < 0 \\ \gamma(a \cup \eta^{4q-2p+2}) \star \eta^{p-2q}, & 2q - p \geq 0. \end{cases}$$

Theorem 5. *The morphism ψ induces an isomorphism*

$$\tilde{\psi}: BO^{*,*}(X)[\eta^{-1}] \xrightarrow{\sim} W^{*,*}(X).$$

Proof. First of all we need to check that ψ is a homomorphism of bigraded algebras. One can easily see that it is agreed with the gradings and additive. In order to check the multiplicativity take $a \in BO^{p,q}(X), b \in BO^{r,s}(X)$. The case of $2q - p < 0, 2r - s < 0$ follows from Lemma 6:

$$\begin{aligned} \psi(a \cup b) &= \gamma(a \cup b) \star \eta^{p+r-2(q+s)} = (-1)^{p(r-s)} \gamma(a) \star \gamma(b) \star \eta^{p+r-2(q+s)} = \\ &= (-1)^{p(r-s)} (-1)^{(r-s)(p-2q)} \gamma(a) \star \eta^{p-2q} \star \gamma(b) \star \eta^{r-2s} = \psi(a) \star \psi(b). \end{aligned}$$

The rest cases are quite the same and straightforward, so we leave the detailed check to the reader.

By Lemma 7 we know that

$$\psi(\eta) = \gamma(\eta) \star \eta = -\eta$$

thus $\psi(\eta)$ is invertible and ψ induces a homomorphism

$$\tilde{\psi}: BO^{*,*}(X)[\eta^{-1}] \rightarrow W^{*,*}(X).$$

At last note that $\psi: BO^{p,q}(X) \rightarrow W^{p,q}(X)$ is an isomorphism for $2q - p < 0$, hence $\tilde{\psi}$ is an isomorphism as well. □

7. A MOTIVIC VARIANT OF A THEOREM BY CONNER AND FLOYD.

For this section fix the canonical morphisms

$$\varphi^{Sp}: MSp \rightarrow BO, \quad \psi^{Sp}: MSp \rightarrow MSL$$

given by Theorem 2 and an arbitrary morphism

$$\varphi^{SL}: MSL \rightarrow BO$$

given by Theorem 3.

Definition 14. Let A be a commutative ring spectrum in $\mathcal{SH}(k)$. Define

$$\begin{aligned} A_\eta^{*,*}(Y) &= A^{*,*}(Y) \otimes_{A^{*,*}(pt)} A^{*,*}(pt)[\eta^{-1}], \\ A_\eta^{i,j}(Y) &= (A^{*,*}(Y) \otimes_{A^{*,*}(pt)} A^{*,*}(pt)[\eta^{-1}])^{i,j} \end{aligned}$$

to be the $(1, 1)$ -periodic ring cohomology theory obtained by inverting the stable Hopf map $\eta \in A^{-1,-1}(pt)$.

Lemma 8. *For every small pointed motivic space Y the above morphisms $\varphi^{Sp}, \varphi^{SL}, \psi^{Sp}$ induce ring homomorphisms fitting in the commutative triangle*

$$\begin{array}{ccc} MSp_\eta^{*,*}(Y) & \xrightarrow{\varphi_{\eta,Y}^{Sp}} & BO_\eta^{*,*}(Y) \\ \psi_{\eta,Y}^{Sp} \downarrow & \nearrow \varphi_{\eta,Y}^{SL} & \\ MSL_\eta^{*,*}(Y) & & \end{array}$$

Proof. By the uniqueness part of Theorem 2 it follows that $\varphi^{Sp} = \varphi^{SL}\psi^{Sp}$, so the following triangle is commutative.

$$\begin{array}{ccc} MSp(Y)^{*,*} & \xrightarrow{\varphi_Y^{Sp}} & BO^{*,*}(Y) \\ \psi_Y^{Sp} \downarrow & \nearrow \varphi_Y^{SL} & \\ MSL^{*,*}(Y) & & \end{array}$$

The morphisms φ^{Sp}, ψ^{Sp} are homomorphisms of the monoids, thus φ_Y^{Sp} and ψ_Y^{Sp} are homomorphisms of $\pi^{*,*}(pt)$ -algebras. The last morphism φ_Y^{SL} is a $\pi^{*,*}(pt)$ -algebra homomorphism as well by Lemma 3, so the claim follows via localization. \square

Definition 15. For a ring cohomology theory $A^{*,*}(-)$ and a motivic space Y put

$$A^{4*,2*}(Y) = \bigoplus_{n \in \mathbb{Z}} A^{4n,2n}(Y).$$

Recall the following theorem reconstructing hermitian K -theory via algebraic symplectic cobordism [PW10d, Theorem 1.1].

Theorem 6. *For every small pointed motivic space Y there exists a natural isomorphism*

$$\overline{\varphi}_Y^{Sp}: MSp^{*,*}(Y) \otimes_{MSp^{4*,2*}(pt)} BO^{4*,2*}(pt) \xrightarrow{\cong} BO^{*,*}(Y).$$

induced by φ^{Sp} .

Corollary 2. *For every smooth variety X there exists a natural isomorphism*

$$MSp^{*,*}(X)/(\eta - 1) \otimes_{MSp^{4*,2*}(pt)} W^{2*}(pt) \xrightarrow{\cong} W^*(X).$$

Proof. Theorem 5 combined with Theorem 6 provides an isomorphism

$$\overline{\varphi}_X^{Sp} : MSP_\eta^{*,*}(X) \otimes_{MSP^{4*,2*}(pt)} BO^{4*,2*}(pt) \xrightarrow{\sim} W^*(X)[\eta, \eta^{-1}].$$

Consider the following commutative diagram.

$$\begin{array}{ccc} MSP^{*,*}(X)/(\eta-1) \otimes_{MSP^{4*,2*}(pt)} BO^{4*,2*}(pt) & \xrightarrow{\varphi_2} & W^*(X) \\ \pi \downarrow & \nearrow \varphi_1 & \\ MSP^{*,*}(X)/(\eta-1) \otimes_{MSP^{4*,2*}(pt)} W^{2*}(pt) & & \end{array}$$

Here φ_1, φ_2 and π are induced by φ^{Sp} and the natural surjection

$$BO^{4*,2*}(pt) \cong GW^{2*}(pt) \rightarrow W^{2*}(pt).$$

The map φ_2 is obtained from the isomorphism $\overline{\varphi}_X^{Sp}$ via setting $\eta-1=0$, thus it is an isomorphism as well. The homomorphism π is surjective, hence φ_1 is an isomorphism. \square

The goal of this section is to replace symplectic cobordisms in the above isomorphisms with the special linear ones. Shortening the notation, set

$$\overline{MSL}_\eta^{*,*}(Y) = MSL_\eta^{*,*}(Y) \otimes_{MSL^{4*,2*}(pt)} BO_\eta^{4*,2*}(pt).$$

By Lemma 8, for every small pointed motivic space Y there is a natural homomorphism

$$\overline{\varphi}_{\eta,Y}^{SL} : \overline{MSL}_\eta^{*,*}(Y) \rightarrow BO_\eta^{*,*}(Y).$$

Lemma 9. *For every small pointed motivic space Y there is a natural homomorphism of $\pi^{*,*}(pt)$ -algebras*

$$t_Y : BO_\eta^{*,*}(Y) \rightarrow \overline{MSL}_\eta^{*,*}(Y)$$

such that

- (1) $\overline{\varphi}_{\eta,Y}^{SL} \circ t_Y = id.$
- (2) $t_Y(a) = 1 \otimes a$ for every $a \in BO_\eta^{4*,2*}(pt).$
- (3) $t_{Th(E)}(th^{BO}(\mathcal{T})) = th^{MSL}(\mathcal{T}) \otimes 1$ for every smooth variety X and every special linear bundle $\mathcal{T} = (E, \lambda)$ such that there exists a symplectic form ϕ on E compatible with trivialization λ .

Proof. There is a natural commutative diagram of the following form.

$$\begin{array}{ccc} MSP_\eta^{*,*}(Y) \otimes_{MSP^{4*,2*}(pt)} BO^{4*,2*}(pt) & \xrightarrow{\overline{\varphi}_{\eta,Y}^{Sp}} & BO_\eta^{*,*}(Y) \\ \theta_Y \downarrow & \nearrow \overline{\varphi}_{\eta,Y}^{SL} & \\ MSL_\eta^{*,*}(Y) \otimes_{MSL^{4*,2*}(pt)} BO_\eta^{4*,2*}(pt) & & \end{array}$$

Here θ_Y is induced by ψ_Y^{Sp} . Theorem 6 provides that $\overline{\varphi}_{\eta,Y}^{Sp}$ is an isomorphism, thus we can take $t_Y = \theta_Y \circ (\overline{\varphi}_{\eta,Y}^{Sp})^{-1}$. The first property is clear. The second property follows from the surjectivity of the natural map $BO^{4*,2*}(pt) \rightarrow BO_\eta^{4*,2*}(pt).$

For the third property recall that θ_Y and φ_Y^{Sp} map Thom classes of symplectic bundles to the corresponding Thom classes, so

$$\begin{aligned} t_{Th(E)}(th^{BO}(\mathcal{T})) &= t_{Th(E)}(th^{BO}(E, \phi)) = \theta_Y(th^{MSp}(E, \phi) \otimes 1) = \\ &= th^{MSL}(E, \phi) \otimes 1 = th^{MSL}(\mathcal{T}) \otimes 1. \quad \square \end{aligned}$$

Now we restrict our attention to the indices $(4*, 2*)$, special linear Grassmannians SGr and corresponding Thom spaces. Recall the following "symplectic principle" for the special linear bundles (see [An12, Theorem 7]).

Theorem 7. *Let $\mathcal{T} = (E, \lambda)$ be a special linear bundle of even rank over a smooth variety X . Then there exists a morphism of smooth varieties $p: Y \rightarrow X$ such that $MSL_\eta^{*,*}(Y)$ is a free $MSL_\eta^{*,*}(X)$ -module (via p^{MSL_η}) and p^*E has a canonical symplectic form ϕ compatible with the trivialization $p^*\lambda$.*

Lemma 10. *For the homomorphism*

$$t_{Th(2n, m)}: BO_\eta^{4*, 2*}(Th(2n, m)) \rightarrow \overline{MSL}_\eta^{4*, 2*}(Th(2n, m))$$

we have

$$t_{Th(2n, m)}(th^{BO}(\mathcal{T}(2n, m))) = th^{MSL}(\mathcal{T}(2n, m)) \otimes 1.$$

Proof. Put $\mathcal{T}(2n, m) = (E, \lambda)$. Theorem 7 provides a morphism of smooth varieties $p: Y \rightarrow SGr(2n, m)$ such that p^{MSL_η} is injective and there exists a symplectic form ϕ on p^*E agreed with $p^*\lambda$.

Consider the following diagram.

$$\begin{array}{ccc} \overline{MSL}_\eta^{4*, 2*}(Th(p^*E)) & \xleftarrow{t_{Th(p^*(E))}} & BO_\eta^{4*, 2*}(Th(p^*(E))) \\ \tilde{p}^{MSL_\eta} \uparrow & & \uparrow \tilde{p}^{BO_\eta} \\ \overline{MSL}_\eta^{4*, 2*}(Th(E)) & \xleftarrow{t_{Th(E)}} & BO_\eta^{4*, 2*}(Th(E)) \end{array}$$

Here $\tilde{p}: Th(p^*E) \rightarrow Th(E)$ is induced by the morphism p . Naturality of t yields that the diagram is commutative. Hence, by functoriality of Thom classes and Lemma 9, we obtain

$$\begin{aligned} p^{MSL_\eta} t_{Th(E)}(th^{BO}(\mathcal{T}(2n, m))) &= t_{Th(p^*E)} \tilde{p}^{BO_\eta}(th^{BO}(\mathcal{T}(2n, m))) = \\ &= t_{Th(p^*E)}(th^{BO}(p^*E, p^*\lambda)) = t_{Th(p^*E)}(th^{BO}(p^*E, \phi)) = \\ &= th^{MSL_\eta}(p^*E, \phi) \otimes 1 = th^{MSL_\eta}(p^*\mathcal{T}(2n, m) \otimes 1) = \\ &= p^{MSL_\eta}(th^{MSL}(\mathcal{T}(2n, m)) \otimes 1). \end{aligned}$$

The claim follows since p^{MSL_η} is injective. \square

Theorem 8. *For every small pointed motivic space Y there exists a natural isomorphism*

$$\overline{\varphi}_{\eta, Y}^{SL}: MSL_\eta^{*,*}(Y) \otimes_{MSL_\eta^{4*, 2*}(pt)} BO_\eta^{4*, 2*}(pt) \xrightarrow{\sim} BO_\eta^{*,*}(Y).$$

Proof. First we focus on the indices $(4*, 2*)$. By Lemma 9 it follows that the homomorphism

$$\overline{\varphi}_{\eta, Y}^{SL}: \overline{MSL}_{\eta}^{4*, 2*}(Y) \rightarrow BO_{\eta}^{4*, 2*}(Y)$$

is surjective. In order to check that its section t_Y is surjective as well take an arbitrary element $\alpha \otimes b \in \overline{MSL}_{\eta}^{4*, 2*}(Y)$. One may assume that $\alpha = \beta \cup \eta^{-n}$ for some $n \in \mathbb{N}$ and $\beta \in MSL^{4m-n, 2m-n}(Y) \cong MSL^{4m, 2m}(Y \wedge T^{\wedge n})$. By a similar argument as in Lemma 3 we may assume that

$$\Sigma_T^{2i} \Sigma^{n, n} \beta = g^{MSL} th^{MSL}(\mathcal{T}(2m + 2i, (2m + 2i)^2))$$

for some $i, j \in \mathbb{N}$ and $g \in Hom_{H_{\bullet}(k)}((\Sigma^{n, n} Y) \wedge T^{\wedge 2i}, Th(2m + 2i, (2m + 2i)^2))$. Set $r = 2m + 2i$ and consider the following commutative diagram.

$$\begin{array}{ccc} \overline{MSL}_{\eta}^{4*+4i+n, 2*+2i+n}(Th(r, r^2)) & \xleftarrow{t_{Th}} & BO_{\eta}^{4*+4i+n, 2*+2i+n}(Th(r, r^2)) \\ \downarrow g^{MSL_{\eta}} \otimes id & & \downarrow g^{BO_{\eta}} \\ \overline{MSL}_{\eta}^{4*+4i+n, 2*+2i+n}(\Sigma_T^{2i} \Sigma^{n, n} Y) & \xleftarrow{t_{\Sigma_T^{2i} \Sigma^{n, n} Y}} & BO_{\eta}^{4*+4i+n, 2*+2i+n}(\Sigma_T^{2i} \Sigma^{n, n} Y) \\ \uparrow \Sigma_T^{2i} \Sigma^{n, n} \otimes id \simeq & & \uparrow \Sigma_T^{2i} \Sigma^{n, n} \simeq \\ \overline{MSL}_{\eta}^{4*, 2*}(Y) & \xleftarrow{t_Y} & BO_{\eta}^{4*, 2*}(Y). \end{array}$$

By Lemma 10 and the above considerations we have

$$\begin{aligned} t_Y((\Sigma_T^{2i} \Sigma^{n, n} \otimes id)^{-1} g^{BO_{\eta}}(th^{BO}(\mathcal{T}(r, r^2)) \cup \eta^{-n} \cup b)) &= \\ = (\Sigma_T^{2i} \Sigma^{n, n} \otimes id)^{-1} (g^{MSL_{\eta}} \otimes 1) t_{Th}(th^{BO}(\mathcal{T}(r, r^2)) \cup \eta^{-n} \cup b) &= \\ = (\Sigma_T^{2i} \Sigma^{n, n} \otimes id)^{-1} (g^{MSL_{\eta}} \otimes 1) (th^{MSL}(\mathcal{T}(r, r^2)) \cup \eta^{-n} \otimes b) &= \\ = \beta \cup \eta^{-n} \otimes b = \alpha \cup b. \end{aligned}$$

Thus the section

$$t_Y: BO_{\eta}^{4*, 2*}(Y) \rightarrow \overline{MSL}_{\eta}^{4*, 2*}(Y)$$

is surjective and $\overline{\varphi}_{\eta, Y}^{SL}$ is an isomorphism for the indices $(4*, 2*)$.

In order to obtain the isomorphism for the arbitrary indices (p, q) one should use the suspension isomorphisms

$$\overline{MSL}_{\eta}^{p, q}(Y) \cong \overline{MSL}_{\eta}^{2q, q}(Y \wedge S_s^{2q-p}), \quad BO_{\eta}^{p, q}(Y) \cong BO_{\eta}^{2q, q}(Y \wedge S_s^{2q-p})$$

in case of $2q - p > 0$ and the analogous ones with $\wedge S_s^{2q-p}$ replaced by $\wedge G_m^{\wedge p-2q}$ in case of $p - 2q > 0$. \square

Corollary 3. *For every smooth variety X there exist a natural isomorphism*

$$MSL^{*, *}(X)/(\eta - 1) \otimes_{MSL^{4*, 2*}(pt)} W^{2*}(pt) \xrightarrow{\cong} W^*(X).$$

Proof. The claim follows from Theorems 5 and 8 by the similar reasoning as in Corollary 2. \square

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